On the Generic Representation of Matrix Condensation Processes and Its Consequences

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Abstract

Condensation methods for evaluating the determinant of a square matrix over the years had been restricted to that of Chio's condensation method (CCM) and Dodgson condensation method (DCM); however, recently (Ufuoma, 2016, 2019) introduced a new method of condensation known as the intermediate condensation method (ICM). This method was later reinvented by (Salihu and Marevci, 2019). Thus, in this study, a generic representation of condensation methods is given, which consequently yields a new condensation method for evaluating the determinant of a square matrix which we shall call Okokwu condensation method (OCM).

Keywords and Phrase: Dodgson's condensation, determinant, matrix, Chio's condensation

1. BACKGROUND TO THE STUDY

The two determinantal identities constructed in the mid nineteenth century are due to Chiò's and Dodgson's, from the perspective of their origins in earlier work by (Sylvester, 1851) and (Jacobi, 1841). A good number of proofs of Chiò's and Dodgson's identities are cited for completeness purposes. In a recent review of Roger Hart's book, *The Chinese Roots of Linear Algebra*, the reviewer, Joseph F. Grcar, asserts that many authors including Charles Dodgson have reinvented Chiò's method to evaluate determinants (Joseph, 2012). Although Dodgson's and Chiò's identities considered as methods have many attributes in common, they are fundamentally different. From the historical origins of each identity, explicitly we may link Dodgson's identity to a theorem of Jacobi, and Chiò's identity to a theorem of Sylvester.

The subject of determinant is quite far reaching, cutting across every facet and branch of mathematics, which is quite intriguing and interesting. In Control Theory and Stability Analysis, it is employed in the Routz Harwuz theory and aid the process that produces the characteristic equation that yields the eigen values and its corresponding eigen vectors; in Linear Algebra, it is used to establish the existence of solution of linear system; in Differential Equation, it is used to establish the independency of set of solution of a differential equation via the Wronskian's determinant; in Co-ordinate Geometry, it is used to obtain the volume of parallelepiped and the area of a plane figure (Salihu and Marevci, 2021); in Numerical Analysis and Interpolation Theory, it is used to establish the existence of interpolating function via the Vandemond's determinant, to mention but a few. Based on the applicable importance of determinant as highlighted above and many more it is the interest of this research work to look into the method of determinant evaluation known as Condensation method, with particular interest on the method of proof for the case of intermediate condensation method which is already in existence considering the fact that it is the most recent development in this line of research as touching condensation methods.

2. STATEMENT OF THE PROBLEM

It is observed from literature that in recent years much attention has been given to the development of efficient methods of evaluating the determinant of a square matrix via condensation process (Sylvester, 1851; Jacob, 1841; Chio, 1853; Dodgson, 1866; Muir, 1911a, 1911b; Ufuoma, 2016, 2019; Salihu and Marevci, 2019; Harwood, et al., 2016). Condensation methods for evaluating the determinant of a square matrix over the years had been restricted to that of Chio's condensation method (CCM) and Dodgson condensation method (DCM); however, recently (Ufuoma, 2016, 2019) introduced a new method of condensation known as the intermediate condensation method (ICM). This method was later reinvented by (Salihu and Marevci, 2019). Thus, in this study, a new condensation method is proposed for evaluating the determinant of a square matrix which we shall call Okokwu condensation method (OCM).

3. **THE PROPOSED OKOKWU CONDENSATION METHOD (OCM)**

In order to catch a glimpse of the method we wish to propose, we shall introduce a pair-wise ordered multiplication process for the row and column entries as follows:

Let $r_i \in R_i$ be points of the row entries and $c_i \in C_i$ be points of the column entries, then we represent any pair-wise multiplication (cell representation) of the row by column as such

$$\begin{pmatrix} r_i \\ r_{i+1} \end{pmatrix} \times (c_j \quad c_{j+1}) := \left[\begin{Vmatrix} a_{ij} & a_{ij+1} \\ a_{i+1j} & a_{i+1j+1} \end{Vmatrix}\right] \tag{3.1}$$

So that in general, by this representation (definition) we shall have that

$$\begin{pmatrix} R_i \\ R_{i+1} \end{pmatrix} \times (C_j \quad C_{j+1}) := \left(\begin{vmatrix} a_{ij} & a_{ij+1} \\ a_{i+1j} & a_{i+1j+1} \end{vmatrix}\right)_{ij=1}^{n-1} \tag{3.2}$$

Where we consequently define a *condensation condition equation* for new rows as

$$\frac{a_{ij}}{a_{i+1j}} = \frac{R_i}{R_{i+1}} \tag{3.3}$$

Which can also be defined column-wise as will be seen in the sequel.

Now, observe that if we fix row $R_i = R_1 \ \forall\, i$ and column $C_j = C_1 \ \forall\, j$ simultaneously. Then we shall have

$$\begin{pmatrix} R_1 \\ R_{i+1} \end{pmatrix} \times (C_1 \quad C_{j+1}) := \left(\begin{vmatrix} a_{11} & a_{1j+1} \\ a_{i+11} & a_{i+1j+1} \end{vmatrix} \right)_{ij=1}^{n-1} \tag{3.4}$$

Furthermore, if we fix only column $C_j = C_1 \ \forall\, j$. Then we shall have

$$\begin{pmatrix} R_i \\ R_{i+1} \end{pmatrix} \times (C_1 \quad C_{j+1}) := \left(\begin{vmatrix} a_{i1} & a_{ij+1} \\ a_{i+11} & a_{i+1j+1} \end{vmatrix} \right)_{ij=1}^{n-1} \tag{3.5}$$

it is interesting to know that with appropriate normalization constant for each of these representations, we shall have (3.2) to yield the Dodgson's condensation, (3.4) to yield the Chio's condensation and finally (3.5) to yield the Ufuoma's (Intermediate) condensation.

We have so far exploit and discussed various methods of evaluating the determinant of large matrices, via the method of Chiò, Dodgson, and Ufuoma with the aim of introducing a new condensation method of equivalent computational complexity in this chapter.

Now, recall in chapter three of this thesis, from equation (3.3) we have

$$\begin{pmatrix} R_i \\ R_{i+1} \end{pmatrix} \times (C_j \quad C_{j+1}) := \left(\begin{vmatrix} a_{ij} & a_{ij+1} \\ a_{i+1j} & a_{i+1j+1} \end{vmatrix} \right)_{ij=1}^{n-1}$$

By fixing row $R_i = R_1 \ \forall\, i$ and column $C_j = C_1 \ \forall\, j$ simultaneously we obtained the representation in equation (3.4) given by

$$\begin{pmatrix} R_1 \\ R_{i+1} \end{pmatrix} \times (C_1 \quad C_{j+1}) := \left(\begin{vmatrix} a_{11} & a_{1j+1} \\ a_{i+11} & a_{i+1j+1} \end{vmatrix} \right)_{ij=1}^{n-1}$$

Which yield the Chio's condensation,

Furthermore, if we fix only column $C_j = C_1 \ \forall\, j$. Then we obtained the representation in equation (3.5) given by

$$\begin{pmatrix} R_i \\ R_{i+1} \end{pmatrix} \times (C_1 \quad C_{j+1}) := \left(\begin{vmatrix} a_{i1} & a_{ij+1} \\ a_{i+11} & a_{i+1j+1} \end{vmatrix} \right)_{ij=1}^{n-1}$$

Which yield the Ufuoma's (Intermediate) condensation.

For the purpose of this research we proceed to give a proof of obtaining the determinant of n-square matrix for a new representation by fix only row $R_i = R_1 \; \forall \, i$ in equation (3.3) so that we have

$$\begin{pmatrix} R_1 \\ R_{i+1} \end{pmatrix} \times (C_j \quad C_{j+1}) := \left(\begin{vmatrix} a_{1j} & a_{1j+1} \\ a_{i+1j} & a_{i+1j+1} \end{vmatrix} \right)_{ij=1}^{n-1} \tag{3.6}$$

We now state the theory for this new condensation known as Okokwu condensation method (OC) or New Intermediate condensation method (NIC) as follows.

Theorem 3.1 (Okokwu condensation). Let A_n be an $n \times n$ matrix (i.e. $A_n = \left(a_{ij}\right)_{n\times n}$), then the (first) Okokwu's condensations of the matrix A_n, is an $(n-1) \times (n-1)$ matrix defined by

$$OC(A_n) = A_{n-1}$$

$$= \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{2(n-1)} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{3(n-1)} & a_{3n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{4(n-1)} & a_{4n} \end{vmatrix} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{n2} & a_{n3} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{n3} & a_{n4} \end{vmatrix} & \ddots \; \cdots & \begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{n(n-1)} & a_{nn} \end{vmatrix} \end{vmatrix}_{(n-1)\times(n-1)}$$

where

$$\det(A_n) = \frac{\det(A_{n-1})}{a_{12} \times a_{13} \times \cdots \times a_{1(n-2)} \times a_{1(n-1)}} = \frac{\det(A_{n-1})}{\prod_{j=2}^{n-1} a_{1j}}; \;\; a_{1j} \neq 0 \; \forall \, j$$

The above process is continued until we obtain a 2×2 matrix whose determinant is easily determined.

Proof

The *condensation condition equation* defined column-wise (for our new colums) is

$$\frac{a_{1j}}{a_{1j+1}} = \frac{C_j}{C_{j+1}} \tag{3.7}$$

$$\Rightarrow a_{1j}C_{j+1} = a_{1j+1}C_j$$

$$\Rightarrow \; a_{1j}C_{j+1} - a_{1j+1}C_j = 0$$

$$\Rightarrow -a_{1j+1}\left(C_j - \frac{a_{1j}}{a_{1j+1}}C_{j+1}\right) = 0 \;\; or \; a_{1j}\left(C_{j+1} - \frac{a_{1j+1}}{a_{1j}}C_j\right) = 0$$

$$\Rightarrow -\left(C_j - \frac{a_{1j}}{a_{1j+1}}C_{j+1}\right) = 0 \;\; or \; \left(C_{j+1} - \frac{a_{1j+1}}{a_{1j}}C_j\right) = 0$$

Now we define the new j^{th} column C_j^{new} for the condensation process as follows

$$C_j^{new} = \begin{cases} -\left(C_j - \dfrac{a_{1j}}{a_{1j+1}} C_{j+1}\right) & ; \ j = 1 \\ \left(C_{j+1} - \dfrac{a_{1j+1}}{a_{1j}} C_j\right) & ; \ j = 2,3,\cdots,n-1 \end{cases} \tag{3.7}$$

So that by applying (3.7), it follows that

$$\left| -\left(C_1 - \frac{a_{1,1}}{a_{12}} C_2\right) \quad C_{j+1} - \frac{a_{1j+1}}{a_{1j}} C_j \right|_{(j=2,3,\cdots,n-1)} = \begin{vmatrix} -\left(C_1 - \dfrac{a_{1,1}}{a_{12}} C_2\right) \\ C_3 - \dfrac{a_{13}}{a_{12}} C_2 \\ C_4 - \dfrac{a_{14}}{a_{13}} C_3 \\ \vdots \\ C_n - \dfrac{a_{1n}}{a_{1n-1}} C_{n-1} \end{vmatrix}^T =$$

$$\left| -\left(C_1 - \frac{a_{1,1}}{a_{12}} C_2\right) \quad C_3 - \frac{a_{13}}{a_{12}} C_2 \quad C_4 - \frac{a_{14}}{a_{13}} C_3 \quad \cdots \quad C_n - \frac{a_{1n}}{a_{1n-1}} C_{n-1} \right|$$

$$= \begin{vmatrix} -\left(a_{21} - \dfrac{a_{1,1}}{a_{12}} a_{22}\right) & a_{23} - \dfrac{a_{13}}{a_{12}} a_{22} & a_{24} - \dfrac{a_{14}}{a_{13}} a_{23} & \dots & a_{2n} - \dfrac{a_{1n}}{a_{1n-1}} a_{2(n-1)} \\ -\left(a_{31} - \dfrac{a_{1,1}}{a_{12}} a_{23}\right) & a_{33} - \dfrac{a_{13}}{a_{12}} a_{32} & a_{34} - \dfrac{a_{14}}{a_{13}} a_{33} & \dots & a_{3n} - \dfrac{a_{1n}}{a_{1n-1}} a_{3(n-1)} \\ -\left(a_{41} - \dfrac{a_{1,1}}{a_{12}} a_{24}\right) & a_{43} - \dfrac{a_{13}}{a_{12}} a_{42} & a_{44} - \dfrac{a_{14}}{a_{13}} a_{43} & \dots & a_{4n} - \dfrac{a_{1n}}{a_{1n-1}} a_{4(n-1)} \\ \vdots & \vdots & \vdots & \vdots \ \ddots & \vdots \\ -\left(a_{n1} - \dfrac{a_{1,1}}{a_{12}} a_{2n}\right) & a_{n3} - \dfrac{a_{13}}{a_{12}} a_{n2} & a_{n4} - \dfrac{a_{14}}{a_{13}} a_{n3} & \dots & a_{nn} - \dfrac{a_{1n}}{a_{1n-1}} a_{n(n-1)} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \begin{vmatrix} \dfrac{-(a_{21}a_{12} - a_{1,1}a_{22})}{a_{12}} & \dfrac{a_{23}a_{12} - a_{13}a_{22}}{a_{12}} & \dfrac{a_{24}a_{13} - a_{14}a_{23}}{a_{13}} & \cdots & \dfrac{a_{2n}a_{1n-1} - a_{1n}a_{2(n-1)}}{a_{1n-1}} \\ \dfrac{-(a_{31}a_{12} - a_{1,1}a_{23})}{a_{12}} & \dfrac{a_{33}a_{12} - a_{13}a_{32}}{a_{12}} & \dfrac{a_{34}a_{13} - a_{14}a_{33}}{a_{13}} & \cdots & \dfrac{a_{3n}a_{1n-1} - a_{1n}a_{3(n-1)}}{a_{1n-1}} \\ \dfrac{-(a_{41}a_{12} - a_{1,1}a_{24})}{a_{12}} & \dfrac{a_{43}a_{12} - a_{13}a_{42}}{a_{12}} & \dfrac{a_{44}a_{13} - a_{14}a_{43}}{a_{13}} & \cdots & \dfrac{a_{4n}a_{1n-1} - a_{1n}a_{4(n-1)}}{a_{1n-1}} \\ \vdots & \vdots & \vdots & \vdots \ \ddots & \vdots \\ \dfrac{-(a_{n1}a_{12} - a_{1,1}a_{2n})}{a_{12}} & \dfrac{a_{n3}a_{12} - a_{13}a_{n2}}{a_{12}} & \dfrac{a_{n4}a_{13} - a_{14}a_{n3}}{a_{13}} & \cdots & \dfrac{a_{nn}a_{1n-1} - a_{1n}a_{n(n-1)}}{a_{1n-1}} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \begin{vmatrix} \dfrac{1}{a_{12}}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \dfrac{1}{a_{12}}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \dfrac{1}{a_{13}}\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \dots & \dfrac{1}{a_{1n-1}}\begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{2(n-1)} & a_{2n} \end{vmatrix} \\ \dfrac{1}{a_{12}}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \dfrac{1}{a_{12}}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \dfrac{1}{a_{13}}\begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} & \dots & \dfrac{1}{a_{1n-1}}\begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{3(n-1)} & a_{3n} \end{vmatrix} \\ \dfrac{1}{a_{12}}\begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} & \dfrac{1}{a_{12}}\begin{vmatrix} a_{12} & a_{13} \\ a_{42} & a_{43} \end{vmatrix} & \dfrac{1}{a_{13}}\begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix} & \dots & \dfrac{1}{a_{1n-1}}\begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{4(n-1)} & a_{4n} \end{vmatrix} \\ \vdots & \vdots & \vdots & \vdots \ \ddots & \vdots \\ \dfrac{1}{a_{12}}\begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \dfrac{1}{a_{12}}\begin{vmatrix} a_{12} & a_{13} \\ a_{n2} & a_{n3} \end{vmatrix} & \dfrac{1}{a_{13}}\begin{vmatrix} a_{13} & a_{14} \\ a_{n3} & a_{n4} \end{vmatrix} & \dots & \dfrac{1}{a_{1n-1}}\begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{n(n-1)} & a_{nn} \end{vmatrix} \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \frac{1}{a_{12}} \times \frac{1}{a_{12}} \times \frac{1}{a_{13}} \cdots$$

$$\times \frac{1}{a_{1n-1}}\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{2(n-1)} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{33} & a_{34} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{3(n-1)} & a_{3n} \end{vmatrix} \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{43} & a_{44} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{4(n-1)} & a_{4n} \end{vmatrix} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \begin{vmatrix} a_{11} & a_{12} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{n2} & a_{n3} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{n3} & a_{n4} \end{vmatrix} & \ddots & \begin{vmatrix} a_{1n-1} & a_{1n} \\ a_{n(n-1)} & a_{nn} \end{vmatrix} \end{vmatrix}_{(n-1)\times(n-1)}$$

Thus

$$|A_n| = a_{12}\left(\frac{1}{a_{12}} \times \frac{1}{a_{12}} \times \frac{1}{a_{13}} \cdots \times \frac{1}{a_{1n-1}}|A_{n-1}|\right) = \frac{1}{a_{12}} \times \frac{1}{a_{13}} \cdots \times \frac{1}{a_{1n-1}}|A_{n-1}|$$

Which is equivalent to

$$\det(A_n) = \frac{\det(A_{n-1})}{a_{12} \times a_{13} \times \cdots \times a_{1(n-2)} \times a_{1(n-1)}} = \frac{\det(A_{n-1})}{\prod_{j=2}^{n-1} a_{1j}};\ \ a_{1j} \neq 0\ \forall\ j$$

As such, for the rest of the condensation we have

$$\det(A_n) = \frac{\det(A_{n-1})}{\pi_1};\ \pi_1 = \prod_{i=2}^{n-1} a_{1j}^1\ \ a_{1j}^1 \neq 0\ \forall\ j$$

$$\det(A_{n-1}) = \frac{\det(A_{n-2})}{\pi_2};\ \pi_2 = \prod_{j=2}^{n-2} a_{1j}^2\ \ a_{1j}^2 \neq 0\ \forall\ j$$

$$\det(A_{n-2}) = \frac{\det(A_{n-3})}{\pi_3};\ \pi_3 = \prod_{j=2}^{n-3} a_{1j}^3\ \ a_{1j}^3 \neq 0\ \forall\ j$$

$$\vdots$$

$$\det(A_{n-k}) = \frac{\det(A_{n-k-1})}{\pi_{k+1}};\ \pi_{k+1} = \prod_{j=2}^{n-k-1} a_{1j}^{k+1}\ \ a_{1j}^{k+1} \neq 0\ \forall\ j,k$$

Now, if we stop at the kth condensation which is assumed to be the last condensation, then we must have that $n-k-1=2$, which implies that $k=n-3$. Then it follows that the last kth condensation is equivalent to

$$\det(A_3) = \frac{\det(A_2)}{\pi_{n-2}};\ \pi_{n-2} = \prod_{j=2}^{2} a_{1j}^{n-2} = a_{1j}^{n-2} \neq 0\ \forall\ j$$

So that taking all together we have

$$\det(A_n) = \frac{1}{\pi_1} \times \frac{1}{\pi_2} \times \frac{1}{\pi_3} \times \cdots \times \frac{\det(A_2)}{\pi_{n-2}} = \frac{\det(A_2)}{\prod_{p=1}^{n-2} \pi_p};\ \ \pi_p \neq 0\ \forall\ p$$

This complete the proof.

We state the following corollary which shows the relationship between Ufuoma condensation method and the newly introduced Okokwu condensation method (OCM).

Corollary 3.2. Let A_n be an $n \times n$ matrix (i.e. $A_n = \left(a_{ij}\right)_{n\times n}$), then the followings relation are equivalent for Okokwu's condensation (OC) and Ufuoma's condensation (UC) of the matrix A_n.

$i)\ OC(A_n) = UC(A_n^T)$

$ii)\ OC(A_n^T) = UC(A_n)$

Example 3.3 Determine the determinant of the given matrices by Okokwu method

1) $\begin{pmatrix} 5 & 3 & 9 \\ 2 & 4 & 11 \\ 1 & 6 & 8 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & 2 & -1 & 0 \\ 0 & 2 & 1 & 5 \\ -3 & 3 & 1 & 2 \end{pmatrix}$

Solution

(1)

$$\textbf{Step1}: \frac{OC\{A_3\}}{\prod_{j=2}^{3-1} a_{1j}^1} = \frac{\begin{pmatrix} 5 & 3 & 9 \\ 2 & 4 & 11 \\ 1 & 6 & 8 \end{pmatrix}}{a_{12}^1} = \frac{OC\left\{\begin{pmatrix} 5 & 3 & 9 \\ 2 & 4 & 11 \\ 1 & 6 & 8 \end{pmatrix}\right\}}{3} = \frac{\begin{pmatrix} \begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 9 \\ 4 & 11 \end{vmatrix} \\ \begin{vmatrix} 5 & 3 \\ 1 & 6 \end{vmatrix} & \begin{vmatrix} 3 & 9 \\ 6 & 8 \end{vmatrix} \end{pmatrix}}{3} = \frac{\begin{pmatrix} 14 & -3 \\ 27 & -30 \end{pmatrix}}{3}$$

$$\textbf{Step 2}: \frac{OC\left\{\begin{pmatrix} 14 & -3 \\ 27 & -30 \end{pmatrix}\right\}}{3\ \prod_{j=2}^{2-1} a_{1j}^2} = \frac{\begin{vmatrix} 14 & -3 \\ 27 & -30 \end{vmatrix}}{3} = \frac{-420 + 81}{3} = -113$$

(2)

$$\textbf{Step1}: \frac{OC\{A_4\}}{\prod_{j=2}^{4-1} a_{1j}^1} = \frac{OC\left\{\begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & 2 & -1 & 0 \\ 0 & 2 & 1 & 5 \\ -3 & 3 & 1 & 2 \end{pmatrix}\right\}}{a_{12}^1 \times a_{13}^1} = \frac{\begin{pmatrix} \begin{vmatrix} 1 & -2 \\ 4 & 2 \end{vmatrix} & \begin{vmatrix} -2 & 3 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} \\ \begin{vmatrix} 1 & -2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix} & \begin{vmatrix} -2 & 3 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} \end{pmatrix}}{-2 \times 3}$$

$$= \frac{\begin{pmatrix} 10 & -4 & 1 \\ 2 & -8 & 14 \\ -3 & -11 & 5 \end{pmatrix}}{-6}$$

$$\textbf{Step2}: \frac{OC\left\{\begin{pmatrix} 10 & -4 & 1 \\ 2 & -8 & 14 \\ -3 & -11 & 5 \end{pmatrix}\right\}}{-6\ \prod_{j=2}^{3-1} a_{1j}^2} = \frac{\begin{pmatrix} \begin{vmatrix} 10 & -4 \\ 2 & -8 \end{vmatrix} & \begin{vmatrix} -4 & 1 \\ -8 & 14 \end{vmatrix} \\ \begin{vmatrix} 10 & -4 \\ -3 & -11 \end{vmatrix} & \begin{vmatrix} -4 & 1 \\ -11 & 5 \end{vmatrix} \end{pmatrix}}{-6a_{12}^2} = \frac{\begin{pmatrix} -72 & -48 \\ -122 & -9 \end{pmatrix}}{-6 \times (-4)} = \frac{\begin{pmatrix} -72 & -48 \\ -122 & -9 \end{pmatrix}}{24}$$

$$\textbf{Step 3}: \frac{OC\left\{\begin{pmatrix} -72 & -48 \\ -122 & -9 \end{pmatrix}\right\}}{24\ \prod_{j=2}^{2-1} a_{1j}} = \frac{\begin{vmatrix} -72 & -48 \\ -122 & -9 \end{vmatrix}}{24} = \frac{648 + 5856}{24} = \frac{-5208}{24} = -217$$

We wish to recommend that this method of Okokwu's condensation be employed over Dodgson's condensation in the computation of determinant of square matrices owing to its minimal computational complexity as will be demonstrated in our next research work. Finally, For the purpose of feature research in this condensation processes we formulate the following unification of the processes considered so far. Let the ordered pair $(p,s),(q,t),(\alpha,\beta)$ be such that

$$|p-q| = \alpha\ ;\ \Rightarrow p-q=\alpha\ or\ p-q=-\alpha\ \ ;\ \Rightarrow p=q+\alpha\ or\ q=p+\alpha$$

$$|s-t| = \beta\ ;\ \Rightarrow s-t=\beta\ or\ s-t=-\beta\ \ ;\ \Rightarrow s=t+\beta\ or\ t=s+\beta$$

Where $p,s,q,t,\alpha,\beta\ \in\ [n]$.

In particular for $q = p+\alpha$ and $t = s+\beta$ let

$$\begin{pmatrix} R_p \\ R_{p+\alpha} \end{pmatrix} \times (C_s \quad C_{s+\beta}) := \left(\begin{vmatrix} a_{ps} & a_{p,s+\beta} \\ a_{p+\alpha,s} & a_{p+\alpha,s+\beta} \end{vmatrix} \right)_{p,s,\alpha,\beta\geq 1} \tag{3.8}$$

Observe that by equating any two of the running variables p,s,α,β equals to one, we obtain various representations of the condensation processes discussed above. Thus

$$\begin{pmatrix} R_p \\ R_{p+\alpha} \end{pmatrix} \times (C_s \quad C_{s+\beta}) := \begin{cases} Chio\ ; p = s = 1 \\ Dodgson\ ; \alpha = \beta = 1 \\ Ufuoma\ ; \alpha = s = 1 \\ Okokwu\ ; p = \beta = 1 \end{cases}$$

The above representation in equation (3.8) which is the generic representation is of more general than the one in equation (3.2). By this, the following research question need to be considered in the feature; i) the implication of this representation if the running variable are not uniformly equal to one? ii) are there other prescribed values for the running variables other than one that could lead to successful evaluation of the determinant of a square matrix minimal computational complexity? iii) if the answers to these questions above in i) and ii) are in the affirmative sense, then what are the theoretical implications of the results?

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