Dodgson's Condensation Iterative Scheme for Evaluating Determinant of Square Matrix

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Abstract

Dodgson's condensation method has remained one of the remarkable methods employed for evaluating determinant of matrices of large order, and for solving large system of linear equation based on its suitability in combining with Cramer's method. In the course of evaluating determinant using Dodgson's condensation for a very large matrix one may run out of track in identifying the interior of the matrix that serve as a divisor for a particular condensed matrix. It is important to keep a check on this happenstance during computation by devising a suitable iteration scheme that implores Dodgson's condensation processes so as to do away with the concept of Duplex Fraction recently introduced by a researcher in a bid to compute determinant of matrices with large order. Thus, the iteration scheme proposed generalizes this concept naturally and ultimately yields the value of the required determinant of the matrix without performing any elementary matrix operations on the row or column of the matrix.

Keywords and Phrase: Dodgson's condensation, determinant, matrix, Duplex Fraction, order of matrix

1. Introduction and Definitions

Over the years, determinant of a matrix has remained a useful tool in mathematics and other related fields of study. The stability of a mathematical model, by using differential equations, is determined by the behaviour of the eigen values of the fundamental (resolvent) matrix. Undoubtedly, these eigen values are obtained by finding the determinant of the fundamental matrix. Moreover, determinant is used to establish the linear independency of the column vectors of a square matrix and linear independency of set of solution to n order ordinary differential equations. Geometrically, the row entries of a matrix of order 3 represent the edges of a parallelopiped in Euclidean space. Thus, the volume of such parallelopiped is the determinant of its edges.

Finding the determinant of a square matrix is one of the prime topics in

Linear Algebra. Many methods for computing the determinants of square matrices include Sarrus' rule and Triangle's rule for matrices of order 3, Cofactor's method, Chio's condensation method and Dodgson's condensation method.

Let $M_n(R)$ be the set of square matrices of order n , so that if $A_n \in M_n(R)$ then

$$A_n = \begin{pmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \vdots & \vdots & \vdots \ \ddots \ \vdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{pmatrix}$$

Definition 3.1 A determinant of A_n (see (Hamiti, 2002; Barnard and Child, 1959; Scott, 1904; Ferrar, 1957) is

$$\det(A_n) = |A_n| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots a_{3n} \\ \vdots & \vdots & \vdots \ \ddots \ \vdots \\ a_{n1} & a_{n2} & a_{n3} \cdots a_{nn} \end{vmatrix} = \sum_{j_1 j_2 j_3 \cdots j_n \in S_n} \mu_{j_1 j_2 j_3 \cdots j_n} a_{1j_1} a_{2j_2} a_{3j_3} \cdots a_{nj_n} \quad (1.1)$$

Summing over the permutation set (symmetric permutation group) S_n, where

$$\mu_{j_1 j_2 j_3 \cdots j_n} = \begin{cases} +1; \ if \ j_1 j_2 j_3 \cdots j_n \ is \ an \ even \ permutation \\ -1; \ if \ j_1 j_2 j_3 \cdots j_n \ is \ an \ odd \ permutation \end{cases}$$

It is also possible to expand a determinant along a row or column using Laplace's formula, which is efficient for relatively small matrices. Which is given by

$$\det(A_n) = \begin{cases} \sum_{j=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \ \ \forall \ i \ (row \ wise) \\ \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \ \ \forall \ j \ (column \ wise) \end{cases} . \quad (1.2)$$

Where the C_{ij} represent the matrix cofactors, i.e., C_{ij} is $(-1)^{i+j}$ times the minor M_{ij} which is the determinant of the matrix that results from A_n by removing the ith row and the jth column, and n is the size of the matrix.

2. MATRIX CONDENSATION PROCESS

To evaluate a determinant, Chiò's and Dodgson's identities both condense an $n \times n$ determinant in a sequence of steps so that ultimately only the calculation of a 2×2 determinant is required. Assuming the entries of the original determinant are integers, in both identities at each step of the reduction, all entries in the determinant remain integers. Non-zero pivots must be chosen at each step in the reduction. Chiò's and Dodgson's determinantal identities, derived from quite different sources, often have been treated as the same identity perhaps because they are both "condensa-tion" methods. Computationally, they both reduce the evaluation of an n×n determinant to the computation of a 2×2 determinant. As an efficient computational method, Chiò's algorithm is clearly the better of the two. For hand calculation of the determinant of a matrix of order greater than three, either of the two is much easier to carry out than the standard Laplacian method. In large numerical calculations where Gaussian elimination is the algorithm of choice, evaluating a determinant requires $\frac{1}{3}n^3$ computations. Using either Dodgson's or Chiò's algorithm requires cn^3 computations, where the constant c is larger for Dodgson's method (not including any row or column exchanges) than for Chiò's. In a nonparallel lsetting the computational complexity of each, O(n^3), is the same as the computational complexity of Gaussian elimination. Both methods can be implemented profitably in a parallel setting. However, the two determinantal identities constructed in the mid nineteenth century, Chiò's and Dodgson's, from the perspective of their origins has been linked to the earlier work (Sylvester, 1851) and (Jacob, 1841). However, we choose to investigate on the iteration scheme for Dodgson's condensation process.

2.1 Dodgson's Condensation Method

In 1866, an English writer Charles Lutwidge Dodgson popularly known as Lewis Carroll (1832-1898) gave a method of computing the determinant of a square matrix by condensation. The method proves to be effective as well as minimizes errors before arriving at the solution (Leggett, Perry, and Torrence, 2009). Dodgson condensation reduces the matrix into 2×2 submatrices for easy computation of determinants. The method reduces the risk of miscalculation as it is bound to divide the determinant of the submatrices by interior elements (Abeles, 1986). The fatal of Dodgson's condensation defect is that the determinant of an interior matrix must not be zero because dividing the determinant of the minors by zero makes the solution indeterminate (Abeles, 1994). The advantage of Dodgson condensation is that the determinant of a square matrix is a rational function of all its connected minors of any two consecutive sizes (Schmidt and Greene, 2011). The fatal defect of Dodgson condensation has a remedy like row (column) permutations, though it may not always work if there are many zero entries in the matrix or the determinant of interior matrix zero – this can happen even if no zeroes appear in the interior of the matrix (Abeles, 2008; Robbins, 2005). In Dodgson's condensation, each smaller matrix contains the 2×2 connected minors of the previous iteration's matrix. The 2×2 connected minors are the determinants of each 2×2 submatrices consisting of adjacent elements of the larger matrix. Beginning with the second stage of iteration, each of these minors is divided by their central element from two stages previous. In this case, Dodgson suggested replacing the zero element with a nonzero element of the matrix by rotating columns or rows and then proceeding with condensation. If all elements of the matrix are zero, then the matrix is trivial, and its determinant is zero. For a given $n \times n$ matrix, a *minor* is any $(n-m) \times (n-m)$ matrix formed by deleting m rows and m columns from A. A *complementary minor* is the resulting $m \times m$ matrix diagonally adjacent to the minor matrix while a *consecutive minor* is one in which the remaining rows and columns in the minor were adjacent in the original matrix. *interior* of A is the $(n-2)\times(n-2)$ consecutive minor that results when the first and last rows and columns of matrix A are deleted, see (Abeles, 1986; Rice and Torrence, 2006, 2007).

As it is well-known, Dodgson's condensation process is also a specialized case of Jacob's theorem (Harwood, et al. 2016). The algorithm involved is one that consider computational simplicity, which is achieved by restricting itself entirely to the calculation of 2×2 determinants, which usually consists of the following steps (Harwood, et al. 2016):

1. Use elementary row and column operations to remove all zeros from the interior of A. Here, the interior of an $n\times n$ $(n\geq 3)$ matrix A, or $intA$, is the $(n-2)\times(n-2)$ consecutive minor that results when the first and last rows and columns of matrix A are deleted.

2. Find the 2×2 determinant for every four adjacent terms to form a new $(n-1)\times(n-1)$ matrix A_{n-1}.

3. Repeat this step to produce an $(n-2)\times(n-2)$ matrix, and then divide each term by the corresponding entry in the interior of the original matrix A, to obtain matrix A_{n-2}.

4. Continue "condensing" the matrix down, until a single number $A_1$is obtained. This final number will be the determinant of $|A_n|$.

Theorem 3 (Dodgson's condensation theorem) *Let A be an* $n\times n$ *matrix. After ksuccessful condensation, Dodgson produces the matrix*

$$A^{(n-k)}=\begin{bmatrix} |A_{1,\cdots,k+1,\ 1,\cdots,k+1}| & |A_{1,\cdots,k+1,\ 2,\cdots,k+2}| & \cdots & |A_{1,\cdots,k+1,\ n-k,\cdots,n}| \\ |A_{2,\cdots,k+2,\ 1,\cdots,k+1}| & |A_{2,\cdots,k+2,\ 2,\cdots,k+2}| & \cdots & |A_{2,\cdots,k+2,\ n-k,\cdots,n}| \\ \vdots & \vdots & \ddots & \vdots \\ |A_{n-k,\cdots,n,\ 1,\cdots,k+1}| & |A_{n-k,\cdots,n,\ 2,\cdots,k+2}| & \cdots & |A_{n-k,\cdots,n,\ n-k,\cdots,n}| \end{bmatrix}$$

Whose entries are the determinants of all $(k+1)\times(k+1)$ *contiguous submatrices of* A. So that in particular if $k=1$, we have the following corollary

Corollary 3.2. (Dodgson's Condensation). Let A_n be an $n\times n$ matrix (i.e. $A_n=(a_{ij})_{n\times n}$), then the (first) Dodgson's condensations of the matrix A_n, is an $(n-1)\times(n-1)$ matrix defined by

$$D(A_n)=A_{n-1}$$

$$=\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \cdots & \begin{vmatrix} a_{1(n-2)} & a_{1(n-1)} \\ a_{2(n-2)} & a_{2(n-1)} \end{vmatrix} & \begin{vmatrix} a_{1(n-1)} & a_{1n} \\ a_{2(n-1)} & a_{2n} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \cdots & \begin{vmatrix} a_{2(n-2)} & a_{2(n-1)} \\ a_{3(n-2)} & a_{3(n-1)} \end{vmatrix} & \begin{vmatrix} a_{2(n-1)} & a_{2n} \\ a_{3(n-1)} & a_{3n} \end{vmatrix} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \begin{vmatrix} a_{(n-2)1} & a_{(n-2)2} \\ a_{(n-1)1} & a_{(n-1)2} \end{vmatrix} & \begin{vmatrix} a_{(n-2)2} & a_{(n-2)3} \\ a_{(n-1)2} & a_{(n-1)3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{(n-2)(n-2)} & a_{(n-2)(n-1)} \\ a_{(n-1)(n-2)} & a_{(n-1)(n-1)} \end{vmatrix} & \begin{vmatrix} a_{(n-2)(n-1)} & a_{(n-2)n} \\ a_{(n-1)(n-1)} & a_{(n-1)n} \end{vmatrix} \\ \begin{vmatrix} a_{(n-1)1} & a_{(n-1)2} \\ a_{n1} & a_{n2} \end{vmatrix} & \begin{vmatrix} a_{(n-1)2} & a_{(n-1)3} \\ a_{n2} & a_{n3} \end{vmatrix} & \cdots & \begin{vmatrix} a_{(n-1)(n-2)} & a_{(n-1)(n-1)} \\ a_{n(n-2)} & a_{n(n-1)} \end{vmatrix} & \begin{vmatrix} a_{(n-1)(n-1)} & a_{(n-1)n} \\ a_{n(n-1)} & a_{nn} \end{vmatrix} \end{vmatrix}_{(n-1)\times(n-1)}$$

Then it follow that

$$A_n(1,1)A_{n-2}(2,2) = A_{n-1}(1,1)A_{n-1}(2,2) - A_{n-1}(2,1)A_{n-1}(1,2)$$

Which implies that the following identity representation of Dodgson's condensation holds,

$$det(A) = det\big(A_n(1,1)\big)$$

$$= \frac{det\big(A_{n-1}(1,1)\big)det\big(A_{n-1}(2,2)\big) - det\big(A_{n-1}(2,1)\big)det\big(A_{n-1}(1,2)\big)}{det\big(A_{n-2}(2,2)\big)} \qquad (2.7)$$

For an $n \times n$ matrix A, let $A_r(i,j)$ denote the r by r minor consisting of r contiguous rows and columns of A, beginning with row i, column j (Amdeberhan and Ekhad, 1997). Note that $A_{n-2}(2,2)$ is the central minor or interior elements; $A_{n-1}(1,1), A_{n-1}(2,2), A_{n-1}(1,2)$ and $A_{n-1}(2,1)$ are the respective northwest, southwest, southeast, northeast, and southwest minors, see (Abeles, 2014; Amdeberhan, 2001; Muir, 1881) and the references therein. According to (Bressoud and Propp, 1999),

"Although the use of division in Dodgson condensation may appear to be a drawback, it serves as a useful form of error checking for calculations done by hand using integer matrices. When the algorithm is carried out correctly, all the entries of all the intervening matrices are integers, making it impossible to know that a mistake has been made when a division does not come out evenly. The approach is helpful for computer calculations as well, particularly".

In the 20th century matrix begins to have some reasonable extent due to its applications in different fields which emerged a new field in mathematics called matrix theory. Since Laplace expansion is a building block for other methods of determinant, only a few of the contributors to the determinant of a matrix in mathematics will be discussed. (Bareiss, 1968) worked on improving the computation of determinants by minimizing the complexity time of the condensation. Although Bareiss algorithm or Montante's method is based on row reduction, it can also be proven using Sylvester's identity(Yap, 2000). The Chinese remainder theorem has been used to compute some cases of determinants (Pan, Yu, and Stewart, 1997; Robbins and Rumsey, 1986) made important studies on the iteration of the Dodgson's Determinantal Identity (DDI) to the discovery of Alternating Sign Matrix Conjecture (ASM). The iteration was from the recurrence of the Laurent polynomials (when $\lambda = -1$) to form lambda determinant of matrix (Mills, Robbins, and Rumsey, 1986). An Alternating Sign Matrix has +1, −1, 0 as an element in every row and column and thus, the ASM conjecture is given as

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)}{(j+n)} \qquad (2.9)$$

Within a decade (Zeilberger, 1997) published a combinatorial proof of DDI. A better algorithm than simple Dodgson's condensation is the recurrence of DDI. Though DDI requires more calculation yet the computational complexity of DDI and Dodgson condensation remain the same (Francisco Neto, 2015; Grcar, 2012) asserted that several authors including Charles Dodgson reinvented Chio's method of evaluating the determinant. However, (Abeles, 2014) stated that Dodgson's identity was a result of a theorem of Jacobi while Chio's identity was from a theorem of Sylvester.

Theorem 5 (Jacobi's theorem on adjoint determinant) *Let A be an* $n \times n$ *matrix, let* (A_{ij}) *be an* $m \times m$ *matrix of A, where* $m < n$*, let* A'_{ij} *be the corresponding* $m \times m$ *minor of* A' *and let* A^*_{ij} *be the complementary* $(n-m) \times (n-m)$ *minor A. Then.*

$$det(A'_{ij}) = (detA)^{m-1} det(A^*_{ij}) \qquad (2.10)$$

By Laplace expansion $A.A' = det\ (A).I$ Thus, $det(A.A') = det(A)det\ (A') = (detA)^n$

It is important to observe that, Dodgson's method is a unique case for both Desanot and Muir's law of extensible minors and Jacobi adjoint matrix theorem. More precisely for Dodgson/Muir determinantal identity is

$$detA = \frac{\sum_{\sigma \in S_k} (-1)^{l(\sigma)} \prod_{j=1}^{k} det(A_{j,\cdots,k+1,\sigma(j),\cdots,k+1})}{det(A_{k+1,\cdots,n,\ k+1\cdots,n})^{k-1}} \qquad (2.11)$$

From the above equation, if $k = 2$ then it turns out to be Dodgson's determinantal identity (DDI). Other special cases where Dodgson's identity was derived are Lagrange, Cauchy and Minding, and Sylvester's identity (Amdeberhan and Ekhad, 1997). It was (Brualdi and Schneider, 1983) that successfully linked Chio and Sylvester's identity by considering Schur's identity as follows:

3. MAIN RESULT

At this point we are now ready to give some important theorems that justify the purpose of this research work as stated earlier before then we consider the definition.

Definition 3.1 Let $(DM)_k(R)$ be the set of square matrix of order k obtained from Dodgson condensation process, if $A_n \in M_n(R)$ with non-zero/non-singular interior points and $B_k \in (DM)_k(R)$ then for every $i, j = 1,2,\cdots,k$

$$\frac{B_k}{int(A_n)} = \left(\frac{b_{ij}}{a_{(n-k-1+i)(n-k-1+j)}}\right)_{ij} \quad iff\ ord\{int(A_n)\} = k;\ det\{int(A_n)\} \neq 0$$

THEOREM 3.2 If $A_n \in M_n(R)$ be a square matrix of order n with non-zero interior points and $D^m(A_n)$ is the mth times application of the Dodgson condensation on A_n is such that for every $j = n-2, n-3, n-4, \cdots, 2, 1$ then the iteration scheme defined by

$$A_{n,j} = \begin{cases} A_{2,1} & if \;\; n = 2 \\ \dfrac{D(A_{n,j+1})}{int(A_{n,j+2})} & if \;\; n \geq 3 \end{cases}$$

Determines the determinant of the matrix A_n as $A_{n,1}$.

Proof: We shall prove this by induction.

Suppose $n = 2$. Then

$$A_{n,j-1} = D(A_{n,j}); \implies A_{2,1} = D(A_{2,2}) = D\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Suppose $n = 3$. Then

$$A_{3,j} = \frac{D(A_{3,j+1})}{int(A_{3,j+2})} \; ; j = 1 \implies A_{3,1} = \frac{D(A_{3,2})}{int(A_{3,3})} = \frac{D\left(D(A_{3,3})\right)}{int(A_{3,3})} = \frac{D^2(A_{3,3})}{int(A_{3,3})}$$

$$= \frac{D^2\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right\}}{int\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right\}} = \frac{D\left\{\begin{pmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{pmatrix}\right\}}{a_{22}} =$$

$$\frac{1}{a_{22}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} = \quad (1.3)$$

$$\frac{1}{a_{22}}\left[\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\right] =$$

$$\frac{1}{a_{22}}[(a_{11}a_{22} - a_{12}a_{21})(a_{22}a_{33} - a_{23}a_{32}) - (a_{12}a_{23} - a_{13}a_{22})(a_{21}a_{32} - a_{22}a_{31})] =$$

$$\frac{1}{a_{22}}\begin{bmatrix} (a_{11}a_{22}a_{22}a_{33} - a_{12}a_{21}a_{22}a_{33} - a_{11}a_{22}a_{23}a_{32} + a_{12}a_{21}a_{23}a_{32}) \\ -(a_{12}a_{23}a_{21}a_{32} - a_{13}a_{22}a_{21}a_{32} - a_{12}a_{23}a_{22}a_{31} + a_{13}a_{22}a_{22}a_{31}) \end{bmatrix} =$$

$$\frac{1}{a_{22}}\begin{bmatrix} a_{11}a_{22}a_{22}a_{33} - a_{12}a_{21}a_{22}a_{33} - a_{11}a_{22}a_{23}a_{32} + a_{12}a_{21}a_{23}a_{32} \\ -a_{12}a_{23}a_{21}a_{32} + a_{13}a_{22}a_{21}a_{32} + a_{12}a_{23}a_{22}a_{31} - a_{13}a_{22}a_{22}a_{31} \end{bmatrix} =$$

$$\frac{1}{a_{22}}\begin{bmatrix} a_{11}a_{22}a_{22}a_{33} - a_{12}a_{21}a_{22}a_{33} - a_{11}a_{22}a_{23}a_{32} \\ +a_{13}a_{22}a_{21}a_{32} + a_{12}a_{23}a_{22}a_{31} - a_{13}a_{22}a_{22}a_{31} \end{bmatrix} =$$

$$\frac{1}{a_{22}}a_{22}\begin{bmatrix} a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \\ +a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} \\ +a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} \end{bmatrix} =$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = |A_3|$$

Which is true

Similarly, suppose $n = 4$. Then

$$A_{4,j} = \frac{D\left(A_{4,j+1}\right)}{int\left(A_{4,j+2}\right)} \; ; j = 2, 1 \quad \Longrightarrow$$

$$A_{4,2} = \frac{D\left(A_{4,3}\right)}{int\left(A_{4,4}\right)} = \frac{D\left(D\left(A_{4,4}\right)\right)}{int\left(A_{4,4}\right)} = \frac{D^2\left(A_{4,4}\right)}{int\left(A_{4,4}\right)}; \; A_{4,1} = \frac{D\left(A_{4,2}\right)}{int\left(A_{4,3}\right)}$$

Now observe that

$$A_{4,2} = \frac{D^2\left(A_{4,4}\right)}{int\left(A_{4,4}\right)} = \frac{D^2\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}\right\}}{int\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}\right\}} = \frac{D\left\{\begin{pmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{pmatrix}\right\}}{\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}}$$

$$=$$

$$\begin{pmatrix} \frac{1}{a_{22}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{23}} \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{32}} \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{33}} \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{pmatrix}$$

Thus, we evaluate the next iteration scheme $A_{4,1}$ given by

$$A_{4,1} = \frac{D\left(A_{4,2}\right)}{int\left(A_{4,3}\right)} = \frac{D\left\{\begin{pmatrix} \frac{1}{a_{22}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{23}} \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{32}} \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{33}} \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{pmatrix}\right\}}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} =$$

$$\frac{1}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \begin{vmatrix} \frac{1}{a_{22}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{23}} \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{32}} \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{33}} \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{vmatrix} \qquad (1.4)$$

$$= \frac{1}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \begin{vmatrix} \dfrac{1}{a_{22}}\left\{a_{22}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}\right\} & \dfrac{1}{a_{23}}\left\{a_{23}\begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix}\right\} \\ \dfrac{1}{a_{32}}\left\{a_{32}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}\right\} & \dfrac{1}{a_{33}}\left\{a_{23}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}\right\} \end{vmatrix} =$$

$$\frac{1}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = |A_{4,1}|$$

Thus, if we continue in this manner and supposed that this is true for $n = r$, then we shall have that

$$A_{r,1} = \frac{1}{\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,r-1} \\ a_{32} & a_{33} & \cdots & a_{3,r-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,2} & a_{r-1,3} & \cdots & a_{r-1,r-1} \end{vmatrix}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,r-1} \\ a_{21} & a_{22} & \cdots & a_{2,r-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,1} & a_{r-1,2} & \cdots & a_{r-1,r-1} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1,r} \\ a_{22} & a_{23} & \cdots & a_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r-1,2} & a_{r-1,3} & \cdots & a_{r-1,r} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{1,r-1} \\ a_{31} & a_{32} & \cdots & a_{2,r-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r,1} & a_{r,2} & \cdots & a_{r,r-1} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,r} \\ a_{32} & a_{33} & \cdots & a_{3,r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r,2} & a_{r,3} & \cdots & a_{r,r} \end{vmatrix} \end{vmatrix}$$

So that for $n = r + 1$, then one deduce that

$$A_{r+1,1} = \frac{1}{\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,r} \\ a_{32} & a_{33} & \cdots & a_{3,r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r,2} & a_{r,3} & \cdots & a_{r,r} \end{vmatrix}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,r} \\ a_{21} & a_{22} & \cdots & a_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r,1} & a_{r,2} & \cdots & a_{r,,r} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1,r+1} \\ a_{22} & a_{23} & \cdots & a_{2,r+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r,2} & a_{r,3} & \cdots & a_{r,r+1} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{1,r} \\ a_{31} & a_{32} & \cdots & a_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r,1} & a_{r,2} & \cdots & a_{r,r} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,r+1} \\ a_{32} & a_{33} & \cdots & a_{3,r+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r+1,2} & a_{r+1,3} & \cdots & a_{r+1,r+1} \end{vmatrix} \end{vmatrix}$$

Thus, we conclude from the iteration scheme that the determinant of the matrix of order n is

$$A_{n,1} = \frac{1}{\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n-1} \\ a_{32} & a_{33} & \cdots & a_{3,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n-1} \end{vmatrix}} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} & \cdots & a_{1,n} \\ a_{22} & a_{23} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,2} & a_{n-1,3} & \cdots & a_{n-1,n} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} & \cdots & a_{1,n-1} \\ a_{31} & a_{32} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,n} \\ a_{32} & a_{33} & \cdots & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,2} & a_{n,3} & \cdots & a_{n,n} \end{vmatrix} \end{vmatrix} \quad (1.5)$$

REMARK 3.3 Observe that equation (1.3), (1.4) and (1.5) are the results obtained in (Farhadian, 2016, 2017), (Farhadian, 2017; Salihu, 2019) and (Salihu, 2012). We shall now proceed to obtain the main results of (Salihu, 2012) for the case of matrices of order five (5) and order six (6) as a special case (corollary) of theorem 3.2 of this research work using elementary row or column operation coupled with the use of Duplex Fraction. In this research work we shall suppose that

$D^m(A_n)$ is the mth times application of the Dodgson condensation (D) on A_n $(n > m)$ and $D_m(A_n)$ is the mth application of the Dodgson condensation on A_n. That is $D^m(A_n) = (D_1 D_2 D_3 \cdots D_m)(A_n)$.

COROLLARY 3.4 Given $A_5 \in M_5(R)$ be a square matrix of order 5 with non-zero interior entries, then for every $j = 3,2,1$ the iteration scheme defined in theorem 3.2 yield the determinant $|A_5|$ as.

$$\frac{a_{33}}{\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix}}
\begin{vmatrix}
\dfrac{\begin{vmatrix}
\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\
\begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix}
\end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^2} &
\dfrac{\begin{vmatrix}
\begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \end{vmatrix} \\
\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix}
\end{vmatrix}}{\begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix}^2} \\
\dfrac{\begin{vmatrix}
\begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \\
\begin{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \\ \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} & \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix}
\end{vmatrix}}{\begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}^2} &
\dfrac{\begin{vmatrix}
\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix} \\
\begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} \end{vmatrix}
\end{vmatrix}}{\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}^2}
\end{vmatrix}$$

Proof.
Since $n = 5$. Then

$$A_{5,j} = \frac{D(A_{5,j+1})}{int(A_{5,j+2})} \; ; j = 3,2,1 \quad \Longrightarrow$$

$$A_{5,3} = \frac{D(A_{5,4})}{int(A_{5,5})} = \frac{D\left(D(A_{5,5})\right)}{int(A_{5,5})} = \frac{D^2(A_{5,5})}{int(A_{5,5})}$$

Thus

$$A_{5,3} = \frac{D^2(A_{5,5})}{int(A_{5,5})} = \frac{D^2\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}\right\}}{int\left\{\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}\right\}} =$$

$$\frac{D\left\{\begin{pmatrix}
\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{14} & a_{15} \\ a_{24} & a_{25} \end{vmatrix} \\
\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\
\begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \\
\begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} & \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix}
\end{pmatrix}\right\}}{\begin{pmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{pmatrix}} =$$

$$\begin{pmatrix} \frac{1}{a_{22}}\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{23}}\begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{24}}\begin{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{32}}\begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{33}}\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{34}}\begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{42}}\begin{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \\ \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} & \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{43}}\begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{44}}\begin{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} \end{vmatrix} \end{pmatrix}$$

Thus, we evaluate the next iteration scheme $A_{5,2}$ given by

$$A_{5,2} = \frac{D(A_{5,3})}{int(A_{5,4})} =$$

$$\frac{D\left\{\begin{pmatrix} \frac{1}{a_{22}}\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{23}}\begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{24}}\begin{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{32}}\begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{33}}\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{34}}\begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{42}}\begin{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \\ \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} & \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{43}}\begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{44}}\begin{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} \end{vmatrix} \end{pmatrix}\right\}}{int\left\{\begin{pmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{14} & a_{15} \\ a_{24} & a_{25} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} & \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} \end{pmatrix}\right\}} =$$

$$\frac{\begin{pmatrix} \begin{vmatrix} \frac{1}{a_{22}}\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{23}}\begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{32}}\begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{33}}\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \frac{1}{a_{23}}\begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{24}}\begin{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{33}}\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{34}}\begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \frac{1}{a_{32}}\begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{33}}\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{42}}\begin{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \\ \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} & \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{43}}\begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \frac{1}{a_{33}}\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{34}}\begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{43}}\begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{44}}\begin{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} \end{vmatrix} \end{vmatrix} \end{pmatrix}}{\begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{pmatrix}}$$

$$=$$

$$\begin{pmatrix} \dfrac{\begin{vmatrix} \frac{1}{a_{22}}\begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{23}}\begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{32}}\begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{33}}\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} & \dfrac{\begin{vmatrix} \frac{1}{a_{23}}\begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{24}}\begin{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{33}}\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{34}}\begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix}} \\ \dfrac{\begin{vmatrix} \frac{1}{a_{32}}\begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{33}}\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{42}}\begin{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \\ \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} & \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{43}}\begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}} & \dfrac{\begin{vmatrix} \frac{1}{a_{33}}\begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{34}}\begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix} \\ \frac{1}{a_{43}}\begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix} & \frac{1}{a_{44}}\begin{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}} \end{pmatrix}$$

$$=$$

$$\begin{pmatrix} \dfrac{\begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^2} & \dfrac{\begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix}^2} \\ \dfrac{\begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \\ \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} & \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}^2} & \dfrac{\begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}^2} \end{pmatrix}$$

Now,

$$A_{5,1} = \frac{D\left(A_{5,2}\right)}{int\left(A_{5,3}\right)} =$$

$$\frac{D\left\{\begin{pmatrix} \dfrac{\begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}^2} & \dfrac{\begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} & \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \\ \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix}^2} \\ \dfrac{\begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \\ \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix} & \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \\ \begin{vmatrix} a_{41} & a_{42} \\ a_{51} & a_{52} \end{vmatrix} & \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix}^2} & \dfrac{\begin{vmatrix} \begin{vmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} \\ \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix} & \begin{vmatrix} a_{24} & a_{25} \\ a_{34} & a_{35} \end{vmatrix} \\ \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} & \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \\ \begin{vmatrix} a_{42} & a_{43} \\ a_{52} & a_{53} \end{vmatrix} & \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} & \begin{vmatrix} a_{34} & a_{35} \\ a_{44} & a_{45} \end{vmatrix} \\ \begin{vmatrix} a_{43} & a_{44} \\ a_{53} & a_{54} \end{vmatrix} & \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix} \end{vmatrix} \end{vmatrix}}{\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix}^2} \end{pmatrix}\right\}}{}$$

$$int\left\{\begin{pmatrix} \frac{1}{a_{22}}\begin{vmatrix}\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}&\begin{vmatrix}a_{12}&a_{13}\\a_{22}&a_{23}\end{vmatrix}\\\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}&\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}\end{vmatrix} & \frac{1}{a_{23}}\begin{vmatrix}\begin{vmatrix}a_{12}&a_{13}\\a_{22}&a_{23}\end{vmatrix}&\begin{vmatrix}a_{13}&a_{14}\\a_{23}&a_{24}\end{vmatrix}\\\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\end{vmatrix} & \frac{1}{a_{24}}\begin{vmatrix}\begin{vmatrix}a_{13}&a_{14}\\a_{23}&a_{24}\end{vmatrix}&\begin{vmatrix}a_{13}&a_{14}\\a_{23}&a_{24}\end{vmatrix}\\\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}&\begin{vmatrix}a_{24}&a_{25}\\a_{34}&a_{35}\end{vmatrix}\end{vmatrix} \\ \frac{1}{a_{32}}\begin{vmatrix}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}&\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}\\\begin{vmatrix}a_{31}&a_{32}\\a_{41}&a_{42}\end{vmatrix}&\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}\end{vmatrix} & \frac{1}{a_{33}}\begin{vmatrix}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\\\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\end{vmatrix} & \frac{1}{a_{34}}\begin{vmatrix}\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}&\begin{vmatrix}a_{24}&a_{25}\\a_{34}&a_{35}\end{vmatrix}\\\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}&\begin{vmatrix}a_{34}&a_{35}\\a_{44}&a_{45}\end{vmatrix}\end{vmatrix} \\ \frac{1}{a_{42}}\begin{vmatrix}\begin{vmatrix}a_{31}&a_{32}\\a_{41}&a_{42}\end{vmatrix}&\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}\\\begin{vmatrix}a_{41}&a_{42}\\a_{51}&a_{52}\end{vmatrix}&\begin{vmatrix}a_{42}&a_{43}\\a_{52}&a_{53}\end{vmatrix}\end{vmatrix} & \frac{1}{a_{43}}\begin{vmatrix}\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\\\begin{vmatrix}a_{42}&a_{43}\\a_{52}&a_{53}\end{vmatrix}&\begin{vmatrix}a_{43}&a_{44}\\a_{53}&a_{54}\end{vmatrix}\end{vmatrix} & \frac{1}{a_{44}}\begin{vmatrix}\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}&\begin{vmatrix}a_{34}&a_{35}\\a_{44}&a_{45}\end{vmatrix}\\\begin{vmatrix}a_{43}&a_{44}\\a_{53}&a_{54}\end{vmatrix}&\begin{vmatrix}a_{44}&a_{45}\\a_{54}&a_{55}\end{vmatrix}\end{vmatrix} \end{pmatrix}\right\}$$

$$=$$

$$\frac{\begin{vmatrix} \dfrac{\begin{vmatrix}\begin{vmatrix}\begin{vmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{vmatrix}&\begin{vmatrix}a_{12}&a_{13}\\a_{22}&a_{23}\end{vmatrix}\\\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}&\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}\end{vmatrix} & \begin{vmatrix}\begin{vmatrix}a_{12}&a_{13}\\a_{22}&a_{23}\end{vmatrix}&\begin{vmatrix}a_{13}&a_{14}\\a_{23}&a_{24}\end{vmatrix}\\\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\end{vmatrix} \\ \begin{vmatrix}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}&\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}\\\begin{vmatrix}a_{31}&a_{32}\\a_{41}&a_{42}\end{vmatrix}&\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}\end{vmatrix} & \begin{vmatrix}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\\\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\end{vmatrix}\end{vmatrix}}{\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}^2} & \dfrac{\begin{vmatrix}\begin{vmatrix}\begin{vmatrix}a_{12}&a_{13}\\a_{22}&a_{23}\end{vmatrix}&\begin{vmatrix}a_{13}&a_{14}\\a_{23}&a_{24}\end{vmatrix}\\\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\end{vmatrix} & \begin{vmatrix}\begin{vmatrix}a_{13}&a_{14}\\a_{23}&a_{24}\end{vmatrix}&\begin{vmatrix}a_{13}&a_{14}\\a_{23}&a_{24}\end{vmatrix}\\\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}&\begin{vmatrix}a_{24}&a_{25}\\a_{34}&a_{35}\end{vmatrix}\end{vmatrix} \\ \begin{vmatrix}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\\\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\end{vmatrix} & \begin{vmatrix}\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}&\begin{vmatrix}a_{24}&a_{25}\\a_{34}&a_{35}\end{vmatrix}\\\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}&\begin{vmatrix}a_{34}&a_{35}\\a_{44}&a_{45}\end{vmatrix}\end{vmatrix}\end{vmatrix}}{\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}^2} \\ \dfrac{\begin{vmatrix}\begin{vmatrix}\begin{vmatrix}a_{21}&a_{22}\\a_{31}&a_{32}\end{vmatrix}&\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}\\\begin{vmatrix}a_{31}&a_{32}\\a_{41}&a_{42}\end{vmatrix}&\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}\end{vmatrix} & \begin{vmatrix}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\\\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\end{vmatrix} \\ \begin{vmatrix}\begin{vmatrix}a_{31}&a_{32}\\a_{41}&a_{42}\end{vmatrix}&\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}\\\begin{vmatrix}a_{41}&a_{42}\\a_{51}&a_{52}\end{vmatrix}&\begin{vmatrix}a_{42}&a_{43}\\a_{52}&a_{53}\end{vmatrix}\end{vmatrix} & \begin{vmatrix}\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\\\begin{vmatrix}a_{42}&a_{43}\\a_{52}&a_{53}\end{vmatrix}&\begin{vmatrix}a_{43}&a_{44}\\a_{53}&a_{54}\end{vmatrix}\end{vmatrix}\end{vmatrix}}{\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}^2} & \dfrac{\begin{vmatrix}\begin{vmatrix}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\\\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\end{vmatrix} & \begin{vmatrix}\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}&\begin{vmatrix}a_{24}&a_{25}\\a_{34}&a_{35}\end{vmatrix}\\\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}&\begin{vmatrix}a_{34}&a_{35}\\a_{44}&a_{45}\end{vmatrix}\end{vmatrix} \\ \begin{vmatrix}\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\\\begin{vmatrix}a_{42}&a_{43}\\a_{52}&a_{53}\end{vmatrix}&\begin{vmatrix}a_{43}&a_{44}\\a_{53}&a_{54}\end{vmatrix}\end{vmatrix} & \begin{vmatrix}\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}&\begin{vmatrix}a_{34}&a_{35}\\a_{44}&a_{45}\end{vmatrix}\\\begin{vmatrix}a_{43}&a_{44}\\a_{53}&a_{54}\end{vmatrix}&\begin{vmatrix}a_{44}&a_{45}\\a_{54}&a_{55}\end{vmatrix}\end{vmatrix}\end{vmatrix}}{\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}^2} \end{vmatrix}}{\frac{1}{a_{33}}\begin{vmatrix}\begin{vmatrix}a_{22}&a_{23}\\a_{32}&a_{33}\end{vmatrix}&\begin{vmatrix}a_{23}&a_{24}\\a_{33}&a_{34}\end{vmatrix}\\\begin{vmatrix}a_{32}&a_{33}\\a_{42}&a_{43}\end{vmatrix}&\begin{vmatrix}a_{33}&a_{34}\\a_{43}&a_{44}\end{vmatrix}\end{vmatrix}}$$

THEOREM 3.5 Given $A_n \in M_n(R)$ be a square matrix of order n with non-zero interior entries and $D^m(A_n)$ is the mth times application of the Dodgson condensation on A_n then for every $j = n-2, n-3, n-4, \cdots, 2, 1$ the iteration scheme defined in theorem 3.2 yield the determinant $|A_n|$ as $A_{n,1}$ given by.

$$A_{n,1} = \frac{D_{n-1}\left(\iddots \, D_5\left(\dfrac{D_4\left(\dfrac{D_3\left(\dfrac{D_2\left(D_1\left(A_{n,n}\right)\right)}{int\left(A_{n,n}\right)} \right)}{int\left(D\left(A_{n,n}\right)\right)} \right)}{int\left(D^2\left(A_{n,n}\right)\right)} \right) \dfrac{\iddots}{int\left(D^3\left(A_{n,n}\right)\right)} \right) \atop \vdots}{int\left(D^{n-3}\left(A_{n,n}\right)\right)}$$

Where $D^m(A_n) = (D_1 D_2 D_3 \cdots D_m)(A_n)$

Proof:

Using the iteration scheme defined in theorem 3,2, it follows that for the case $n-2$ the result is trivial, otherwise

$$A_{n,j} = \frac{D\left(A_{n,J+1}\right)}{int\left(A_{n,J+2}\right)} \; ; n \geq 3 \ \forall \ j = n-2, n-3, n-4, \cdots, 2, 1$$

So that

$$A_{n,n-2} = \frac{D_1(A_{n,n-1})}{int(A_{n,n})} = \frac{D_2\left(D_1(A_{n,n})\right)}{int(A_{n,n})} = \frac{D^2(A_{n,n})}{int(A_{n,n})}$$

$$A_{n,n-3} = \frac{D_3(A_{n,n-2})}{int(A_{n,n-1})} = \frac{D_3(A_{n,n-2})}{int\left(D(A_{n,n})\right)} = \frac{1}{int\left(D(A_{n,n})\right)} D_3\left(\frac{D_2\left(D_1(A_{n,n})\right)}{int(A_{n,n})}\right) = \frac{D_3\left(\frac{D_2\left(D_1(A_{n,n})\right)}{int(A_{n,n})}\right)}{int\left(D(A_{n,n})\right)}$$

$$A_{n,n-4} = \frac{D_4(A_{n,n-3})}{int(A_{n,n-2})} = \frac{D_4(A_{n,n-3})}{int\left(D(A_{n,n-1})\right)} = \frac{D_4(A_{n,n-3})}{int\left(D^2(A_{n,n})\right)} = \frac{D_4\left(\frac{D_3\left(\frac{D_2\left(D_1(A_{n,n})\right)}{int(A_{n,n})}\right)}{int\left(D(A_{n,n})\right)}\right)}{int\left(D^2(A_{n,n})\right)}$$

$\vdots$

$$A_{n,n-k} = \frac{D_k(A_{n,n-k+1})}{int(A_{n,n-k+2})} = \cdots = \frac{D_k\left(\iddots \frac{D_5\left(\frac{D_4\left(\frac{D_3\left(\frac{D_2\left(D_1(A_{n,n})\right)}{int(A_{n,n})}\right)}{int\left(D(A_{n,n})\right)}\right)}{int\left(D^2(A_{n,n})\right)}\right)}{int\left(D^3(A_{n,n})\right)}\right)}{\vdots \atop int\left(D^{k-2}(A_{n,n})\right)}$$

Thus, for $k = n - 1$, we will have

$$A_{n,1} = \frac{D_{n-1}\left(\iddots \frac{D_5\left(\frac{D_4\left(\frac{D_3\left(\frac{D_2\left(D_1(A_{n,n})\right)}{int(A_{n,n})}\right)}{int\left(D(A_{n,n})\right)}\right)}{int\left(D^2(A_{n,n})\right)}\right)}{int\left(D^3(A_{n,n})\right)}\right)}{\vdots \atop int\left(D^{n-3}(A_{n,n})\right)}$$

This completes the Prove. We noticed that theorem 3.5 representation look rather subtle and complex for application purposes, hence we give an equivalent representation of the result in theorem 3.5 which could be written in some worth friendlier manner, we do that in the corollary that follow.

COROLLARY 3.6 Given the result of theorem 3.5 then for every $n \geq 2$ its equivalent representation is given by

$$L_{3,1} = \frac{D_2(L_{2,1})}{int(A_{n,n})} \qquad\qquad L_{3,1} = \frac{D_2(L_{2,1})}{int(L_{1,1})}$$

$$L_{4,1} = \frac{D_3(L_{3,1})}{int\left(D(A_{n,n})\right)} \qquad\qquad L_{4,1} = \frac{D_3(L_{3,1})}{int(L_{2,1})}$$

$$L_{5,1} = \frac{D_4(L_{4,1})}{int\left(D^2(A_{n,n})\right)} \quad \Leftrightarrow \quad L_{5,1} = \frac{D_4(L_{4,1})}{int(L_{3,1})}$$

$$L_{6,1} = \frac{D_5(L_{5,1})}{int\left(D^3(A_{n,n})\right)} \qquad\qquad L_{6,1} = \frac{D_5(L_{5,1})}{int(L_{4,1})}$$

$$\vdots \qquad\qquad \vdots$$

$$L_{n,1} = \frac{D_{n-1}(L_{n-1,1})}{int\left(D^{n-3}(A_{n,n})\right)} = A_{n,1} \qquad L_{n,1} = \frac{D_{n-1}(L_{n-1,1})}{int(L_{n-2,1})} = A_{n,1}$$

where $L_{1,1} = A_{n,n}$, $L_{2,1} = D_1(A_{n,n}) = D(A_{n,n})$, $L_{n-2,1} = D^{n-3}(A_{n,n})$.

We now proceed to apply the result of this corollary 3.6 to specific problems of determinant

4. APPLICATION

Example 3.7 Determine the determinant of the following matrices;

1) $\begin{pmatrix} 5 & 3 & 9 \\ 2 & 4 & 11 \\ 1 & 6 & 8 \end{pmatrix}$ 2) $\begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & 2 & -1 & 0 \\ 0 & 2 & 1 & 5 \\ -3 & 3 & 1 & 2 \end{pmatrix}$

Solution

Using the formula in corollary 3.6, it follows that for $n = 3$ we have

$$L_{3,1} = A_{3,1} = \frac{D_2\left(D_1(A_{n,n})\right)}{int(A_{n,n})} = \frac{D_2\left(D_1\left\{\begin{pmatrix} 5 & 3 & 9 \\ 2 & 4 & 11 \\ 1 & 6 & 8 \end{pmatrix}\right\}\right)}{int\left\{\begin{pmatrix} 5 & 3 & 9 \\ 2 & 4 & 11 \\ 1 & 6 & 8 \end{pmatrix}\right\}} = \frac{D\left\{\begin{pmatrix} \begin{vmatrix} 5 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 9 \\ 4 & 11 \end{vmatrix} \\ \begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix} & \begin{vmatrix} 4 & 11 \\ 6 & 8 \end{vmatrix} \end{pmatrix}\right\}}{4}$$

$$= \frac{D\left\{\begin{pmatrix} 14 & -3 \\ 8 & -34 \end{pmatrix}\right\}}{4} = \frac{-476 + 24}{4} = -113$$

Next, for $n = 4$ we have

$$L_{3,1} = \frac{D_2\left(D_1(A_{n,n})\right)}{int(A_{n,n})}, \quad L_{4,1} = \frac{D_3(L_{3,1})}{int\left(D(A_{n,n})\right)} = A_{4,1}$$

So that

$$L_{3,1} = \frac{D_2\left(D_1(A_{n,n})\right)}{int(A_{n,n})} = \frac{D_2\left(D_1\left\{\begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & 2 & -1 & 0 \\ 0 & 2 & 1 & 5 \\ -3 & 3 & 1 & 2 \end{pmatrix}\right\}\right)}{int\left\{\begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & 2 & -1 & 0 \\ 0 & 2 & 1 & 5 \\ -3 & 3 & 1 & 2 \end{pmatrix}\right\}} =$$

$$\frac{D\left\{\begin{pmatrix} \begin{vmatrix} 1 & -2 \\ 4 & 2 \end{vmatrix} & \begin{vmatrix} -2 & 3 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 3 & 1 \\ -1 & 0 \end{vmatrix} \\ \begin{vmatrix} 4 & 2 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} -1 & 0 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ -3 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 1 & 2 \end{vmatrix} \end{pmatrix}\right\}}{\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}} = \frac{D\left\{\begin{pmatrix} 10 & -4 & 1 \\ 8 & 4 & -5 \\ 6 & -1 & -3 \end{pmatrix}\right\}}{\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}} = \frac{\begin{pmatrix} \begin{vmatrix} 10 & -4 \\ 8 & 4 \end{vmatrix} & \begin{vmatrix} -4 & 1 \\ 4 & -5 \end{vmatrix} \\ \begin{vmatrix} 8 & 4 \\ 6 & -1 \end{vmatrix} & \begin{vmatrix} 4 & -5 \\ -1 & -3 \end{vmatrix} \end{pmatrix}}{\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}}$$

$$= \begin{pmatrix} \frac{\begin{vmatrix} 10 & -4 \\ 8 & 4 \end{vmatrix}}{2} & \frac{\begin{vmatrix} -4 & 1 \\ 4 & -5 \end{vmatrix}}{-1} \\ \frac{\begin{vmatrix} 8 & 4 \\ 6 & -1 \end{vmatrix}}{2} & \frac{\begin{vmatrix} 4 & -5 \\ -1 & -3 \end{vmatrix}}{1} \end{pmatrix} = \begin{pmatrix} 36 & -16 \\ -16 & -17 \end{pmatrix}$$

Then,

$$L_{4,1} = A_{4,1} = \frac{D_3(L_{3,1})}{int\left(D(A_{n,n})\right)} = \frac{D_3(L_{3,1})}{int\left(D(A_{n,n})\right)} = \frac{D_3\left\{\begin{pmatrix} 36 & -16 \\ -16 & -17 \end{pmatrix}\right\}}{int\left(D\left\{\begin{pmatrix} 1 & -2 & 3 & 1 \\ 4 & 2 & -1 & 0 \\ 0 & 2 & 1 & 5 \\ -3 & 3 & 1 & 2 \end{pmatrix}\right\}\right)}$$

$$= \frac{D_3\left\{\begin{pmatrix} 36 & -16 \\ -16 & -17 \end{pmatrix}\right\}}{int\left\{\begin{pmatrix} 10 & -4 & 1 \\ 8 & 4 & -5 \\ 6 & -1 & -3 \end{pmatrix}\right\}} = \frac{\begin{vmatrix} 36 & -16 \\ -16 & -17 \end{vmatrix}}{4} = \frac{-868}{4} = -217$$

5. CONCLUSION

The result of our theorem is not restricted to any particular order of matrix, hence it is applicable to any arbitrary square matrix of order n. Hence the results of our research work generalize some related works in the literature, in particular that of (Farhadian, 2016, 2017; Salihu, 2012, 2019).

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